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On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 1. Fundamental principles

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A new frozen-in field $\boldsymbol{w}$ (generalizing vorticity) is constructed for ideal magnetohydrodynamic flow. In conjunction with the frozen-in magnetic field $\boldsymbol{h}$, this is used to obtain a generalized Weber transformation of the MHD equations, expressing the velocity as a bilinear form in generalized Weber variables. This expression is also obtained from Hamilton's principle of least action, and the canonically conjugate Hamiltonian variables for MHD flow are identified. Two alternative energy-type variational principles for three-dimensional steady MHD flow are established. Both involve a functional R which is the sum of the total energy and another conserved functional, the volume integral of a function Φ of Lagrangian coordinates. It is shown that the first variation $\delta^1 R$ vanishes if Φ is suitably chosen (as minus a generalized Bernoulli integral). Expressions for the second variation $\delta^2 R$ are presented.

1. Introduction

A variational principle that finds its origins in the work of Kelvin (1910) has been formulated and developed, for three-dimensional steady flows of an ideal fluid, by Fjortoft (1950) and Arnold (1965*a*), and has been used in a consideration of the stability of two-dimensional flows to finite-amplitude disturbances by Arnold (1965*b*). The underlying Hamiltonian structure of the Euler equations plays an important role in such an approach, as do any invariants associated with the equations (Holm *et al.* 1985; McIntyre & Shepherd 1987; Vladimirov 1987). The Arnold approach to problems of stability is summarized by Saffman (1992, §14.2). Similar variational principles have been developed for the treatment of the stability of magnetostatic equilibria of perfectly conducting fluids (Bernstein *et al.* 1958) and for the basic characterization of equilibrium states (Woltjer 1958; Taylor 1974; see also the recent monograph of Biskamp 1993, which gives a detailed treatment of these topics).

There are subtle differences between the Euler flow problem and the magnetostatic problem associated with the fact that in the former case it is the curl of the basic velocity field (i.e. the vorticity) which is 'frozen' under unsteady Euler evolution, whereas in the latter case it is the basic field itself (the magnetic field) which is frozen under arbitrary unsteady deformations (the fluid being assumed perfectly conducting). These differences have been discussed by Moffatt (1985, 1986) both in relation to the problem of characterizing, or determining, steady states, and in relation to the stability of these states.

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The problems become more acute when the stability of general solutions of the MHD equations (with non-zero velocity and magnetic field) is considered. In this situation, the magnetic field is still frozen, but the vorticity field is not, the Lorentz force being in general rotational. In order to construct a theory of stability of such states, it is important to identify the field that replaces vorticity as the frozen-in dynamical field, for only then can the counterpart of 'isovortical' perturbations be identified.

Earlier theories (Frieman & Rotenberg 1960; Moffatt 1989; Friedlander & Vishik 1990) have considered virtual displacements under which the magnetic field is frozen, but have not addressed the problem of identifying the second frozen field which (together with the magnetic field) determines the structure and evolution of perturbations from a given steady state. The existence of such a field is suggested by the cross-helicity invariant (the integrated scalar product of velocity and magnetic field) (Woltjer 1958) which is topological in character (Moffatt 1969) and which therefore presumably involves the mutual linkage of two frozen fields; one of these is magnetic field, but what is the other when the rotational Lorentz force is operative?

In §2 of this paper, we identify this second field in the following way. First, we define a solenoidal vector field $\boldsymbol{m}(\boldsymbol{x}, t)$ with the property that the rate of change of the flux of $\boldsymbol{m}$ across any material (Lagrangian) surface element δS is equal to the flux of current $\boldsymbol{j}$ across the same element. We then define the vector field

$$\boldsymbol{w} = \boldsymbol{\omega} + \nabla \times (\boldsymbol{h} \times \boldsymbol{m}), \tag{1.1}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{h}$ are the vorticity and magnetic fields, and we show that $\boldsymbol{w}$ is a frozen-in field.

Recognition of the roles of the fields $\boldsymbol{m}$ and $\boldsymbol{w}$ allows us in §3 to obtain the generalization of Weber's transformation of the Euler equations to the situation when $\boldsymbol{h} \neq 0$ (equation (3.18) below) and, equivalently, the generalization of Cauchy's formula in terms of $\boldsymbol{w}$ instead of $\boldsymbol{\omega}$ (equation (3.2)). It also enables us to demonstrate (§4) that the MHD equations can be obtained from a Hamiltonian variational principle, and that the equations have Hamiltonian structure in which $(\boldsymbol{h}, \boldsymbol{g})$ are canonically conjugate variables, where $\boldsymbol{g}$ is a vector potential for the field $\boldsymbol{m}$.

In §5 we consider properties of steady solutions of the MHD equations augmented by the equation for the auxiliary field $\boldsymbol{m}$, and in §6 we obtain a general energy variational principle for three-dimensional steady MHD flows, which leads to the construction of the appropriate second variation of the energy functional R (equations (6.13)–(6.15)). Finally, in §7, we develop an alternative form of energy variational principle that may be used when the unperturbed velocity field has Clebsch representation (7.11), the streamlines then being closed curves. Again, the second variation of energy is calculated by a clear procedure, and presented in a form (equation (7.39)) that will be useful in subsequent applications.

2. The frozen fields of ideal magnetohydrodynamics

Suppose that an incompressible, inviscid, perfectly conducting fluid of unit density is contained in a domain $\mathscr{D}$ with fixed boundary $\partial\mathscr{D}$. Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the velocity field in the fluid and let $\boldsymbol{h}(\boldsymbol{x}, t)$ be the magnetic field (in Alfvén velocity units). Then

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{h} = 0 \quad \text{in} \quad \mathscr{D}, \tag{2.1}$$

and we adopt the natural boundary conditions

$$\boldsymbol{n} \cdot \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{h} = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{2.2}$$

The fields $\boldsymbol{u}$ and $\boldsymbol{h}$ evolve according to the equations

$$\mathrm{D}\boldsymbol{u} \equiv \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \boldsymbol{j}\times\boldsymbol{h}, \tag{2.3}$$

$$\mathrm{L}\boldsymbol{h} \equiv \boldsymbol{h}_t - \nabla\times(\boldsymbol{u}\times\boldsymbol{h}) = 0, \tag{2.4}$$

where $p(\boldsymbol{x}, t)$ is the pressure field, and

$$\boldsymbol{j} = \nabla\times\boldsymbol{h} \tag{2.5}$$

is the current density. The operator D is the material, or Lagrangian, derivative, the equation $\mathrm{D}f = 0$ implying that f is conserved on particle trajectories. The operator L is a form of Lie derivative, the equation $\mathrm{L}\boldsymbol{h} = 0$ meaning that the $\boldsymbol{h}$-field is 'frozen in the fluid', the flux of $\boldsymbol{h}$ through any closed material circuit being conserved.

Equations (2.3) and (2.4) have three quadratic integral invariants, namely the energy

$$E = \frac{1}{2}\int_{\mathscr{D}} (\boldsymbol{u}^2 + \boldsymbol{h}^2)\,\mathrm{d}\tau, \tag{2.6}$$

the magnetic helicity

$$\mathscr{H}_M = \int_{\mathscr{D}} \boldsymbol{h}\cdot\mathrm{curl}^{-1}\boldsymbol{h}\,\mathrm{d}\tau, \tag{2.7}$$

and the cross-helicity

$$\mathscr{H}_C = \int_{\mathscr{D}} \boldsymbol{u}\cdot\boldsymbol{h}\,\mathrm{d}\tau \tag{2.8}$$

(Woltjer 1958). Conservation of $\mathscr{H}_M$ is associated with the invariance of the topology of the $\boldsymbol{h}$-field (Moffatt 1969); likewise, conservation of $\mathscr{H}_C$ is associated with the fact that, although the flux of vorticity through an arbitrary closed material circuit is not conserved (the Lorentz force $\boldsymbol{j}\times\boldsymbol{h}$ being in general rotational), the flux of vorticity through a closed h-line (which is a *particular* material circuit) *is* conserved.

It will be convenient to adopt the Poisson bracket notation (Arnold 1965)

$$\{\boldsymbol{A}, \boldsymbol{B}\} \equiv \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{2.9}$$

for any two vector fields $\boldsymbol{A}, \boldsymbol{B}$ satisfying the solenoidal conditions

$$\nabla\cdot\boldsymbol{A} = \nabla\cdot\boldsymbol{B} = 0. \tag{2.10}$$

Obviously,

$$\{\boldsymbol{A}, \boldsymbol{B}\} = -\{\boldsymbol{B}, \boldsymbol{A}\}; \tag{2.11}$$

moreover, the Jacobi identity

$$\{\boldsymbol{A}, \{\boldsymbol{B}, \boldsymbol{C}\}\} + \{\boldsymbol{B}, \{\boldsymbol{C}, \boldsymbol{A}\}\} + \{\boldsymbol{C}, \{\boldsymbol{A}, \boldsymbol{B}\}\} = 0 \tag{2.12}$$

may be easily verified. With this notation, (2.4) becomes

$$\boldsymbol{h}_t = \{\boldsymbol{u}, \boldsymbol{h}\}. \tag{2.13}$$

Also, for an arbitrary solenoidal field $\boldsymbol{A}$,

$$\begin{aligned}
\mathrm{L}\{\boldsymbol{h}, \boldsymbol{A}\} &= \frac{\partial}{\partial t}\{\boldsymbol{h}, \boldsymbol{A}\} - \{\boldsymbol{u}, \{\boldsymbol{h}, \boldsymbol{A}\}\} \\
&= \{\boldsymbol{h}, \boldsymbol{A}_t\} + \{\{\boldsymbol{u}, \boldsymbol{h}\}, \boldsymbol{A}\} - \{\boldsymbol{u}, \{\boldsymbol{h}, \boldsymbol{A}\}\} \quad \text{using (2.13)} \\
&= \{\boldsymbol{h}, \boldsymbol{A}_t\} + \{\boldsymbol{h}, \{\boldsymbol{A}, \boldsymbol{u}\}\} \quad \text{using (2.11), (2.12)} \\
&= \{\boldsymbol{h}, \mathrm{L}\boldsymbol{A}\}.
\end{aligned} \tag{2.14}$$

Note further that the curl of (2.3) give the equation for vorticity $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ in the form

$$\boldsymbol{\omega}_t = \{\boldsymbol{u}, \boldsymbol{\omega}\} + \{\boldsymbol{j}, \boldsymbol{h}\}. \tag{2.15}$$

Let us now define an auxiliary dimensionless vector field $\boldsymbol{m}(\boldsymbol{x}, t)$ by the equations

$$\mathrm{L}\boldsymbol{m} \equiv \boldsymbol{m}_t - \{\boldsymbol{u}, \boldsymbol{m}\} = \boldsymbol{j}, \quad \nabla \cdot \boldsymbol{m} = 0. \tag{2.16}$$

Equivalently, if $\delta S(t)$ is any area element moving with the fluid, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{m} \cdot \delta \boldsymbol{S}) = \boldsymbol{j} \cdot \delta \boldsymbol{S}. \tag{2.17}$$

The physical interpretation of $\boldsymbol{m}$ is thus that it is the time-integrated current density across such a (Lagrangian) area element. Note that $\boldsymbol{m}$ is not uniquely determined unless $\boldsymbol{m}(\boldsymbol{x}, 0)$ is specified; however, the properties (2.16) are sufficient for our present purpose.

Let us now define a 'modified velocity field'

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{h} \times \boldsymbol{m}, \tag{2.18}$$

and a corresponding 'modified vorticity field'

$$\boldsymbol{w} = \nabla \times \boldsymbol{v} = \boldsymbol{\omega} + \{\boldsymbol{h}, \boldsymbol{m}\}. \tag{2.19}$$

Then

$$\begin{aligned} \mathrm{L}\boldsymbol{w} &= \mathrm{L}\boldsymbol{\omega} + \mathrm{L}\{\boldsymbol{h}, \boldsymbol{m}\} \\ &= \mathrm{L}\boldsymbol{\omega} + \{\boldsymbol{h}, \mathrm{L}\boldsymbol{m}\} \quad \text{using (2.14)} \\ &= \boldsymbol{\omega}_t - \{\boldsymbol{u}, \boldsymbol{\omega}\} + \{\boldsymbol{h}, \boldsymbol{j}\} \quad \text{using (2.16)} \\ &= 0 \quad \text{using (2.15)}. \end{aligned} \tag{2.20}$$

Hence $\boldsymbol{w}$ is now a frozen field; since $\boldsymbol{w}$ reduces to $\boldsymbol{\omega}$ when $\boldsymbol{h} = 0$, the field $\boldsymbol{w}$ evidently provides the appropriate frozen-field generalization of vorticity when $\boldsymbol{h} \neq 0$. We shall therefore describe $\boldsymbol{w}$ as the *generalized vorticity field*.

We can now give a new and more transparent interpretation of the cross-helicity $\mathscr{H}_C$. For, using (2.18), (2.8) beomes

$$\mathscr{H}_C = \int_{\mathscr{D}} \boldsymbol{v} \cdot \boldsymbol{h} \, \mathrm{d}\tau. \tag{2.21}$$

Since both $\boldsymbol{w}(= \operatorname{curl} \boldsymbol{v})$ and $\boldsymbol{h}$ are now frozen fields, their mutual topology is conserved; for example, the flux of $\boldsymbol{w}$ through any closed $\boldsymbol{h}$-line is conserved. By the arguments of Moffatt (1969, 1981), $\mathscr{H}_C$ provides a measure of the mutual linkage of the fields $\boldsymbol{h}$ and $\boldsymbol{w}$.

Note the trivial solution $\boldsymbol{w} \equiv 0$ of (2.20). This corresponds to $\boldsymbol{v} = \nabla \alpha$ for some scalar field α, or, from (2.18),

$$\boldsymbol{u} = \nabla \alpha + \boldsymbol{m} \times \boldsymbol{h}. \tag{2.22}$$

We may describe this as a *generalized potential flow*.

Finally, note that the essential property $\mathrm{L}\boldsymbol{w} = 0$ is unaffected if we add to $\boldsymbol{m}$ any field $\boldsymbol{m}_1$ satisfying $\mathrm{L}\{\boldsymbol{h}, \boldsymbol{m}_1\} = \{\boldsymbol{h}, \mathrm{L}\boldsymbol{m}_1\} = 0$. Thus, if $\boldsymbol{m} \to \boldsymbol{m} + \boldsymbol{m}_1$ where $\boldsymbol{m}_1$ is any solution of $\mathrm{L}\boldsymbol{m}_1 = \boldsymbol{f}$ with $\{\boldsymbol{h}, \boldsymbol{f}\} = 0$, then $\boldsymbol{w} \to \boldsymbol{w} + \boldsymbol{w}_1$ where $\boldsymbol{w}_1 = \{\boldsymbol{h}, \boldsymbol{m}_1\}$, but

$$\mathrm{L}(\boldsymbol{w} + \boldsymbol{w}_1) = \mathrm{L}\boldsymbol{w} + \mathrm{L}\boldsymbol{w}_1 = 0$$

and the modified field $\boldsymbol{w} + \boldsymbol{w}_1$ is still therefore frozen.

3. Generalized Weber transformation

Suppose now that the fluid particles are labelled by Lagrangian coordinates $\boldsymbol{a} = (a_1, a_2, a_3)$. The label of the particle passing through $\boldsymbol{x}$ at time t is then

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t), \tag{3.1}$$

and
$$\mathrm{D}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{a} = 0. \tag{3.2}$$

The particle paths are given by the inverse function
$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t), \tag{3.3}$$

and the initial position of the particle labelled $\boldsymbol{a}$ is
$$\boldsymbol{x}(\boldsymbol{a}, 0) = \boldsymbol{X}(\boldsymbol{a}), \quad \text{say.} \tag{3.4}$$

A natural choice of label would be $\boldsymbol{a} = \boldsymbol{X}$; however it proves convenient to retain the extra freedom represented by the 'rearrangement function' $\boldsymbol{X}(\boldsymbol{a})$.

The mapping $\boldsymbol{X}(\boldsymbol{a}) \rightarrow \boldsymbol{x}(\boldsymbol{a}, t)$ is induced by the flow
$$\boldsymbol{u} = \mathrm{D}\boldsymbol{x} = (\partial \boldsymbol{x}/\partial t)_{\boldsymbol{a}=\mathrm{const.}}, \tag{3.5}$$

and is a volume-preserving diffeomorphism with the property
$$\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = +1. \tag{3.6}$$

We shall suppose that the mapping $\boldsymbol{a} \rightarrow \boldsymbol{X}(\boldsymbol{a})$ is also a volume-preserving diffeomorphism, so that
$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}\frac{\partial(X_1, X_2, X_3)}{\partial(a_1, a_2, a_3)} = +1 \tag{3.7}$$
also.

We now seek to transform the equation of motion (1.3) to integrable form, by generalization of the argument of Weber (1858) (see, for example, Serrin 1958). First, multiply (1.3) by $\partial x_k/\partial a_i$, giving
$$(\mathrm{D}u_k)\frac{\partial x_k}{\partial a_i} = -\frac{\partial p}{\partial x_k}\frac{\partial x_k}{\partial a_i} - (\boldsymbol{h}\times\boldsymbol{j})_k\frac{\partial x_k}{\partial a_i}. \tag{3.8}$$

Now
$$(\mathrm{D}u_k)\frac{\partial x_k}{\partial a_i} = \mathrm{D}\left(\frac{\partial x_k}{\partial a_i}u_k\right) - \frac{\partial}{\partial a_i}(\tfrac{1}{2}\boldsymbol{u}^2). \tag{3.9}$$

Also we have the identity
$$(\boldsymbol{h}\times\boldsymbol{j})_k\frac{\partial x_k}{\partial a_i} = \mathrm{D}\left\{(\boldsymbol{h}\times\boldsymbol{m})_k\frac{\partial x_k}{\partial a_i}\right\}. \tag{3.10}$$

Hence, using (2.18), (3.8) may be written
$$\mathrm{D}\left(v_k\frac{\partial x_k}{\partial a_i}\right) = \frac{\partial}{\partial a_i}(\tfrac{1}{2}\boldsymbol{u}^2 - p) = \mathrm{D}\left(\frac{\partial \alpha}{\partial a_i}\right), \tag{3.11}$$

where
$$\alpha = \int_0^t (\tfrac{1}{2}\boldsymbol{u}^2 - p)\,|_{\boldsymbol{a}=\mathrm{const.}}\,\mathrm{d}t. \tag{3.12}$$

Integrating along a particle path $\boldsymbol{a} = \text{const.}$, we obtain
$$v_k\frac{\partial x_k}{\partial a_i} = \frac{\partial \alpha}{\partial a_i} + b_i, \tag{3.13}$$

where the constant of integration is given from the initial condition by

$$b_i = V_k(a)\frac{\partial X_k}{\partial a_i}, \quad V(a) = v(a, 0). \tag{3.14}$$

Here $\boldsymbol{b}$, defined in this way as a function of $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$, also satisfies the condition of Lagrangian invariance,

$$\mathrm{D}\boldsymbol{b} = 0. \tag{3.15}$$

In equation (3.13), the terms are best regarded as functions of $\boldsymbol{a}$ and t. We may however immediately convert to Eulerian form by multiplying by $\partial a_i/\partial x_j$, giving

$$\boldsymbol{v} = \nabla\alpha + b_k \nabla a_k, \tag{3.16}$$

in which now α, $\boldsymbol{b}$ and $\boldsymbol{a}$ are regarded as functions of $\boldsymbol{x}$ and t. Here and subsequently we use a mixed vector/suffix notation in which, for example,

$$(b_k \nabla a_k)_i \equiv b_k \frac{\partial}{\partial x_i} a_k, \tag{3.17}$$

the repeated suffix k being summed. From (2.18) and (3.16) it follows immediately that

$$\boldsymbol{u} = \nabla\alpha + \boldsymbol{m}\times\boldsymbol{h} + b_k \nabla a_k. \tag{3.18}$$

Together with the equations

$$\left.\begin{array}{l} \mathrm{D}\boldsymbol{a} = \mathrm{D}\boldsymbol{b} = 0, \quad \mathrm{D}\alpha = \tfrac{1}{2}\boldsymbol{u}^2 - p, \\ \mathrm{L}\boldsymbol{h} = 0, \quad \mathrm{L}\boldsymbol{m} = \boldsymbol{j} \end{array}\right\} \tag{3.19}$$

this provides the required generalization of Weber's (1868) transformation of the governing equations (2.3)–(2.5). Conversely, by applying the operator D to (3.18) and using (3.19), the equation of motion (2.3) is recovered. Equations (3.18) and (3.19) are thus equivalent to (2.3)–(2.5).

The field $\boldsymbol{w}$ is now given by

$$\boldsymbol{w} = \nabla\times\boldsymbol{v} = \nabla b_k \times \nabla a_k, \tag{3.20}$$

a formula that, in conjunction with $\mathrm{D}\boldsymbol{a} = \mathrm{D}\boldsymbol{b} = 0$, again exhibits the frozen-in character of $\boldsymbol{w}$. This result is precisely equivalent to the Cauchy representation

$$w_i(\boldsymbol{x}, t) = w_j(\boldsymbol{a}, 0)\frac{\partial x_i}{\partial a_k}\frac{\partial a_k}{\partial X_j}. \tag{3.21}$$

The Eulerian form (3.20) is however more useful in what follows.

In the special situation when $\boldsymbol{a} = \boldsymbol{X}$, (3.18) and (3.20) reduce to

$$\left.\begin{array}{l} \boldsymbol{u} = \nabla\alpha + \boldsymbol{m}\times\boldsymbol{h} + V_k \nabla X_k, \\ \boldsymbol{w} = \nabla V_k \times \nabla X_k. \end{array}\right\} \tag{3.22}$$

If moreover we adopt initial conditions

$$\boldsymbol{m}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \tag{3.23}$$

then $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \nabla\alpha + \boldsymbol{m}\times\boldsymbol{h} + u_{0k}\nabla X_k, \tag{3.24}$$

the initial conditions for α and $\boldsymbol{X}$ being

$$\alpha(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{X}(\boldsymbol{x}, 0) = \boldsymbol{x}. \tag{3.25}$$

If $\boldsymbol{u}_0(\boldsymbol{x}) \equiv 0$, then (3.24) reduces to the generalized potential flow (2.22).

4. Hamiltonian variational principle and canonically conjugate variables

We now show that equations (3.18), (3.19) may be obtained from a Hamiltonian variational principle. Let

$$\mathscr{L}(\boldsymbol{u},\boldsymbol{h}) = \tfrac{1}{2}(\boldsymbol{u}^2-\boldsymbol{h}^2) \tag{4.1}$$

be the Lagrangian density, and consider the variational problem

$$\delta\int_{\mathscr{D}}\int_0^T \mathscr{L}(\boldsymbol{u},\boldsymbol{h})\,\mathrm{d}\tau\,\mathrm{d}t = 0, \tag{4.2}$$

subject to the constraints

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{u} &= 0 \quad \text{(incompressibility)},\\ \mathrm{L}\boldsymbol{h} &= 0 \quad \text{(frozen-in condition for } \boldsymbol{h}\text{)},\\ \mathrm{D}\boldsymbol{a} &= 0 \quad \text{(identity of particles)}.\end{aligned}\right\} \tag{4.3}$$

Note that $\mathrm{L}\boldsymbol{h} = 0$ implies that $\mathrm{D}(\nabla\cdot\boldsymbol{h}) = 0$, so that $\nabla\cdot\boldsymbol{h} = 0$ should be regarded as an initial condition, and not as an additional constraint. Introducing Lagrange multipliers $\alpha(\boldsymbol{x},t)$, $\boldsymbol{g}(\boldsymbol{x},t)$, $\boldsymbol{b}(\boldsymbol{x},t)$, the variational principle (4.2), (4.3) may be written

$$\delta\int_{\mathscr{D}}\int_0^T (\mathscr{L}+\alpha\nabla\cdot\boldsymbol{u}-\boldsymbol{g}\cdot\mathrm{L}\boldsymbol{h}-\boldsymbol{b}\cdot\mathrm{D}\boldsymbol{a})\,\mathrm{d}\tau\,\mathrm{d}t = 0, \tag{4.4}$$

where now $\boldsymbol{u}$, $\boldsymbol{h}$ and $\boldsymbol{a}$ are to be varied subject only to boundary conditions

$$\boldsymbol{n}\cdot\delta\boldsymbol{u} = \boldsymbol{n}\cdot\delta\boldsymbol{h} = 0 \quad \text{on} \quad \partial\mathscr{D}, \tag{4.5}$$

and initial and final conditions

$$\delta\boldsymbol{u} = \delta\boldsymbol{h} = \delta\boldsymbol{a} = 0 \quad \text{at} \quad t = 0, T. \tag{4.6}$$

If we carry out the variation of (4.4), and integrate by parts as necessary using (4.5) and (4.6), we may then equate to zero the coefficients of the independent variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{h}$, $\delta\boldsymbol{a}$, giving

$$\delta\boldsymbol{u}:\quad \boldsymbol{u} = \nabla\alpha-\boldsymbol{h}\times(\nabla\times\boldsymbol{g})+b_k\nabla a_k, \tag{4.7}$$

$$\delta\boldsymbol{h}:\quad \boldsymbol{h} = \boldsymbol{g}_t-\boldsymbol{u}\times(\nabla\times\boldsymbol{g}), \tag{4.8}$$

$$\delta\boldsymbol{a}:\quad \mathrm{D}\boldsymbol{b} = 0. \tag{4.9}$$

Note that (4.7) is identical with (3.18) provided

$$\boldsymbol{m} = \nabla\times\boldsymbol{g}. \tag{4.10}$$

Moreover the curl of (4.8) then gives immediately $\boldsymbol{j} = \mathrm{L}\boldsymbol{m}$ as in (3.19), and the remaining equations of (3.19) are contained in (4.3) and (4.9). The initial condition $\nabla\cdot\boldsymbol{h} = 0$ at $t = 0$ is satisfied provided

$$\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{g}) = \nabla\cdot(\boldsymbol{u}\times\boldsymbol{m}) \quad \text{at} \quad t = 0. \tag{4.11}$$

Canonically conjugate variables

Consider now the first variation of energy

$$\delta E = \int_{\mathscr{D}} (\boldsymbol{h}\cdot\delta\boldsymbol{h}+\boldsymbol{u}\cdot\delta\boldsymbol{u})\,\mathrm{d}\tau, \tag{4.12}$$

in which $\boldsymbol{u}$ is given by (4.7), but now the independently varied fields are $\boldsymbol{h}, \boldsymbol{g}, \boldsymbol{b}, \boldsymbol{a}$, so that

$$\delta \boldsymbol{u} = \nabla \delta\alpha - \delta \boldsymbol{h} \times (\nabla \times \boldsymbol{g}) - \boldsymbol{h} \times (\nabla \times \delta \boldsymbol{g}) + \delta b_k \nabla a_k + b_k \nabla \delta a_k, \tag{4.13}$$

where $\delta\alpha$ is determined so that $\nabla \cdot \delta \boldsymbol{u} = 0$. Using the conditions (2.1) and (2.2), (4.12) is easily transformed to

$$\delta E = \int_{\mathscr{D}} \{\delta h_i(\boldsymbol{h} + \boldsymbol{u} \times (\nabla \times \boldsymbol{g}))_i - \delta g_i [\nabla \times (\boldsymbol{u} \times \boldsymbol{h})]_i - \delta a_i (\boldsymbol{u} \cdot \nabla) b_i + \delta b_i (\boldsymbol{u} \cdot \nabla) a_i\} \, \mathrm{d}\tau. \tag{4.14}$$

Hence, from (4.3), (4.8) and (4.9), we have immediately

$$\left.\begin{aligned} \partial h_i/\partial t &= [\nabla \times (\boldsymbol{u} \times \boldsymbol{h})]_i = -\delta E/\delta g_i, \\ \partial g_i/\partial t &= h_i + [\boldsymbol{u} \times (\nabla \times \boldsymbol{g})]_i = \delta E/\delta h_i, \end{aligned}\right\} \tag{4.15}$$

$$\left.\begin{aligned} \partial a_i/\partial t &= -(\boldsymbol{u} \cdot \nabla) a_i = -\delta E/\delta b_i, \\ \partial b_i/\partial t &= -(\boldsymbol{u} \cdot \nabla) b_i = \delta E/\delta a_i, \end{aligned}\right\} \tag{4.16}$$

and so $(\boldsymbol{h}, \boldsymbol{g})$ and $(\boldsymbol{a}, \boldsymbol{b})$ are canonically conjugate variables.

5. Steady MHD flows

Consider now steady solutions of (2.3), (2.4) and (2.16)

$$\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}), \quad \boldsymbol{h} = \boldsymbol{H}(\boldsymbol{x}), \quad \boldsymbol{m} = \boldsymbol{M}(\boldsymbol{x}), \quad p = P(\boldsymbol{x}), \tag{5.1}$$

and the associated fields

$$\boldsymbol{\omega} = \boldsymbol{\Omega}(\boldsymbol{x}) = \nabla \times \boldsymbol{U}, \quad \boldsymbol{w} = \boldsymbol{W}(\boldsymbol{x}) = \boldsymbol{\Omega} + \{\boldsymbol{H}, \boldsymbol{M}\}. \tag{5.2}$$

The Lagrangian variables $\boldsymbol{a}, \boldsymbol{b}$ are in general unsteady:

$$\boldsymbol{a} = \boldsymbol{A}(\boldsymbol{x}, t), \quad \boldsymbol{b} = \boldsymbol{B}(\boldsymbol{x}, t), \tag{5.3}$$

where

$$\mathrm{D}_0 \boldsymbol{A} = \mathrm{D}_0 \boldsymbol{B} = 0, \quad \mathrm{D}_0 = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla. \tag{5.4}$$

From (2.20), (2.13) and (2.16), we have

$$\{\boldsymbol{U}, \boldsymbol{W}\} = \{\boldsymbol{U}, \boldsymbol{H}\} = \{\boldsymbol{U}, \boldsymbol{M}\} + \nabla \times \boldsymbol{H} = 0, \tag{5.5}$$

and hence

$$\boldsymbol{U} \times \boldsymbol{W} = \nabla I, \quad \boldsymbol{U} \times \boldsymbol{H} = \nabla J, \quad \boldsymbol{U} \times \boldsymbol{M} + \boldsymbol{H} = \nabla K, \tag{5.6}$$

where I, J and K may be described as generalized Bernoulli functions. Note that

$$\begin{aligned} \nabla I &= \boldsymbol{U} \times \boldsymbol{\Omega} + \boldsymbol{U} \times [\nabla \times (\boldsymbol{H} \times \boldsymbol{M})] \\ &= \nabla(P + \tfrac{1}{2}U^2) - \boldsymbol{H} \times [\nabla \times (\boldsymbol{U} \times \boldsymbol{M})] + \boldsymbol{U} \times [\nabla \times (\boldsymbol{H} \times \boldsymbol{M})] \end{aligned}$$

from (2.3) and (5.4). Hence it may be shown that

$$I = P + \tfrac{1}{2}U^2 + \boldsymbol{M} \cdot (\boldsymbol{U} \times \boldsymbol{H}). \tag{5.7}$$

From (5.5),

$$\boldsymbol{U} \cdot \nabla I = \boldsymbol{U} \cdot \nabla J = 0, \tag{5.8}$$

and so the streamlines lie both on surfaces $I = \text{const.}$ and $J = \text{const.}$, and are therefore the curves of intersection of these surfaces. Hence, unless $\nabla I \equiv 0$ and/or $\nabla J \equiv 0$ in some subdomain of the flow, the streamlines are closed curves. By virtue of (5.8) both I and J may be expressed as functions of the Lagrangian coordinates

$$I = I(\boldsymbol{A}, \boldsymbol{B}), \quad J = J(\boldsymbol{A}, \boldsymbol{B}). \tag{5.9}$$

(There is considerable freedom here since, as noted earlier, $\boldsymbol{B}$ is expressible, via (3.14), as a function of $\boldsymbol{A}$.)

Now, from (3.20), $\boldsymbol{W} = \nabla B_k \times \nabla A_k$, so that the first of (5.6) becomes

$$\boldsymbol{U} \times (\nabla B_k \times \nabla A_k) = \nabla I(\boldsymbol{A}, \boldsymbol{B}),$$

or equivalently

$$\left[(\boldsymbol{U}\cdot\nabla) A_k - \frac{\partial I}{\partial B_k}\right]\frac{\partial B_k}{\partial x_i} - \left[(\boldsymbol{U}\cdot\nabla) B_k + \frac{\partial I}{\partial A_k}\right]\frac{\partial A_k}{\partial x_i} = 0. \tag{5.10}$$

Clearly, this is satisfied provided

$$(\boldsymbol{U}\cdot\nabla) A_k = \frac{\partial I}{\partial B_k}, \quad (\boldsymbol{U}\cdot\nabla) B_k = -\frac{\partial I}{\partial A_k}, \tag{5.11}$$

and we may use the freedom in the specification of the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ to ensure that these equations are satisfied.

Finally note that since $\boldsymbol{U}\cdot\nabla J = \boldsymbol{H}\cdot\nabla J = 0$, the surfaces $J = \text{const.}$ contain both $\boldsymbol{U}$-lines and $\boldsymbol{H}$-lines. In particular, the boundary $\partial\mathscr{D}$ is one such surface, i.e.

$$J = J_0(\text{const.}) \quad \text{on} \quad \partial\mathscr{D}. \tag{5.12}$$

6. A general energy variational principle for 3D steady MHD flows

Consider now a steady MHD flow of the form (5.1), (5.2), satisfying the conditions (5.6). We adopt the generalized Weber representation (3.18), so that

$$\boldsymbol{U}(\boldsymbol{x}) = \nabla\alpha_0 + \boldsymbol{M}\times\boldsymbol{H} + B_k \nabla A_k, \tag{6.1}$$

where α_0 is the field that ensures that

$$\nabla\cdot\boldsymbol{U} = 0 \quad \text{in} \quad \mathscr{D}, \quad \boldsymbol{n}\cdot\boldsymbol{U} = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{6.2}$$

We now consider smooth instantaneous independent variations $\delta\boldsymbol{h}$, $\delta\boldsymbol{m}$, $\delta\boldsymbol{a}$, $\delta\boldsymbol{b}$ of the fields $\boldsymbol{H}$, $\boldsymbol{M}$, $\boldsymbol{A}$, $\boldsymbol{B}$ where $\delta\boldsymbol{h}$, $\delta\boldsymbol{m}$ satisfy

$$\nabla\cdot\delta\boldsymbol{m} = \nabla\cdot\delta\boldsymbol{h} = 0 \quad \text{in} \quad \mathscr{D}, \quad \boldsymbol{n}\cdot\delta\boldsymbol{h} = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{6.3}$$

Let $\delta^1\boldsymbol{u}$ be the corresponding first variation of $\boldsymbol{U}$, given from (3.18) by

$$\delta^1\boldsymbol{u} = \nabla\delta^1\alpha + \boldsymbol{M}\times\delta\boldsymbol{h} + \delta\boldsymbol{m}\times\boldsymbol{H} + \delta b_k \nabla A_k + B_k\nabla\delta a_k, \tag{6.4}$$

where again $\delta^1\alpha$ is chosen so that

$$\nabla\cdot\delta^1\boldsymbol{u} = 0 \quad \text{in} \quad \mathscr{D}, \quad \boldsymbol{n}\cdot\delta^1\boldsymbol{u} = 0 \quad \text{on} \quad \partial\mathscr{D}.$$

Consider now the functional

$$R = E + \int_{\mathscr{D}} \Phi(\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}\tau = \int_{\mathscr{D}} [\tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{h}^2) + \Phi(\boldsymbol{a},\boldsymbol{b})]\,\mathrm{d}\tau, \tag{6.5}$$

where $\Phi(\boldsymbol{a},\boldsymbol{b})$ is an arbitrary function of $\boldsymbol{a},\boldsymbol{b}$ satisfying

$$\mathrm{D}\Phi = \mathrm{D}\boldsymbol{a}\cdot\partial\Phi/\partial\boldsymbol{a} + \mathrm{D}\boldsymbol{b}\cdot\partial\Phi/\partial\boldsymbol{b} = 0. \tag{6.6}$$

This condition guarantees that $\int \Phi(\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}\tau$, and so R, is invariant under the evolution (2.3), (2.4). In the above steady state, R is given by

$$R = R_S = \int [\tfrac{1}{2}(\boldsymbol{U}^2 + \boldsymbol{H}^2) + \Phi(\boldsymbol{A},\boldsymbol{B})]\,\mathrm{d}\tau, \tag{6.7}$$

and its first variation is given by

$$\delta^1 R = \int [\boldsymbol{U}\cdot\delta^1\boldsymbol{u}+\boldsymbol{H}\cdot\delta\boldsymbol{h}+\delta\boldsymbol{a}\cdot\partial\Phi/\partial\boldsymbol{A}+\delta\boldsymbol{b}\cdot\partial\Phi/\partial\boldsymbol{B}]\,\mathrm{d}\tau. \tag{6.8}$$

Substituting for $\delta^1\boldsymbol{u}$ from (6.4) and rearranging, this becomes

$$\delta^1 R = \int \{(\boldsymbol{H}+\boldsymbol{U}\times\boldsymbol{M})\cdot\delta\boldsymbol{h}-(\boldsymbol{U}\times\boldsymbol{H})\cdot\delta\boldsymbol{m} + (-\boldsymbol{U}\cdot\nabla\boldsymbol{B}+\partial\Phi/\partial\boldsymbol{A})\cdot\delta\boldsymbol{a}+(\boldsymbol{U}\cdot\nabla\boldsymbol{A}+\partial\Phi/\partial\boldsymbol{B})\cdot\delta\boldsymbol{b}\}\,\mathrm{d}\tau. \tag{6.9}$$

If we now choose the arbitrary function $\Phi(\boldsymbol{A},\boldsymbol{B})$ to be

$$\Phi(\boldsymbol{A},\boldsymbol{B}) = -I(\boldsymbol{A},\boldsymbol{B}) \tag{6.10}$$

then, by virtue of (5.11), the coefficients of the variations $\delta\boldsymbol{a},\delta\boldsymbol{b}$ vanish. Moreover, using (5.6), (6.9) then reduces to

$$\delta^1 R = \int (\delta\boldsymbol{h}\cdot\nabla K-\delta\boldsymbol{m}\cdot\nabla J)\,\mathrm{d}\tau = \int_{\partial\mathscr{D}} (K\delta\boldsymbol{h}\cdot\boldsymbol{n}-J\delta\boldsymbol{m}\cdot\boldsymbol{n})\,\mathrm{d}S.$$

Hence, since $\delta\boldsymbol{h}\cdot\boldsymbol{n}=0$ and $J=J_0$ (const.) on $\partial\mathscr{D}$,

$$\delta^1 R = -J_0\int_{\partial\mathscr{D}} \boldsymbol{n}\cdot\delta\boldsymbol{m}\,\mathrm{d}S = -J_0\int_{\mathscr{D}} \nabla\cdot\delta\boldsymbol{m}\,\mathrm{d}\tau = 0. \tag{6.11}$$

Hence the functional

$$R = \int [\tfrac{1}{2}(\boldsymbol{u}^2+\boldsymbol{h}^2)-I(\boldsymbol{a},\boldsymbol{b})] \tag{6.12}$$

is stationary under variations about a steady state $\boldsymbol{U}(\boldsymbol{x}),\boldsymbol{H}(\boldsymbol{x})$, provided that $\boldsymbol{m}$, $\boldsymbol{h}$, $\boldsymbol{a}$, $\boldsymbol{b}$ are regarded as the independent variables, $\boldsymbol{u}$ being given by (3.18).

We can now construct the second variation of R under the same conditions; this is

$$\delta^2 R = \int_{\mathscr{D}} \{\tfrac{1}{2}(\delta^1\boldsymbol{u}^2+\delta\boldsymbol{h}^2)+\boldsymbol{U}\cdot\delta^2\boldsymbol{u}-\delta^2 I\}\,\mathrm{d}\tau, \tag{6.13}$$

where

$$\delta^2\boldsymbol{u} = \nabla\delta^2\alpha+\delta\boldsymbol{m}\times\delta\boldsymbol{h}+\delta b_k\nabla\delta a_k \tag{6.14}$$

($\delta^2\alpha$ being chosen so that $\nabla\cdot\delta^2\boldsymbol{u}=0$, $\boldsymbol{n}\cdot\delta^2\boldsymbol{u}=0$ on $\partial\mathscr{D}$), and

$$2\delta^2 I = \frac{\partial^2 I}{\partial A_i\partial A_k}\delta a_i\delta a_k+2\frac{\partial^2 I}{\partial A_i\partial B_k}\delta a_i\delta b_k+\frac{\partial^2 I}{\partial B_i\partial B_k}\delta b_i\delta b_k. \tag{6.15}$$

According to general principles (Arnold 1965*a*), the steady solution $(\boldsymbol{U}(\boldsymbol{x}),\boldsymbol{H}(\boldsymbol{x}))$ is stable if $\delta^2 R$ is definite in sign for all variations $(\delta\boldsymbol{h},\delta\boldsymbol{m},\delta\boldsymbol{a},\delta\boldsymbol{b})$ satisfying (6.3). It is difficult however to use this condition, because of the dependence of the integrand in (6.13) on the time-dependent fields $\boldsymbol{A}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$. We seek an alternative simpler form of variational principle in the next section.

7. Alternative form of energy variational principle for 3D steady MHD flows

Let $a(\boldsymbol{x},t)$ be a single generalized Lagrangian coordinate satisfying

$$\mathrm{D}a = 0, \tag{7.1}$$

and let $b(\boldsymbol{x}, t)$ now be defined by

$$b(\boldsymbol{x}, t) = (\boldsymbol{w}\cdot\nabla)\,a, \tag{7.2}$$

with $\boldsymbol{w}$ still defined by (2.19). In view of the conditions (2.20) and (7.1), b is also constant on particle paths, i.e.

$$\mathrm{D}b = 0. \tag{7.3}$$

In considering steady solutions (5.1), (5.2), we may now suppose that a and b are also steady, i.e.

$$a = A(\boldsymbol{x}), \quad b = B(\boldsymbol{x}), \tag{7.4}$$

where

$$(\boldsymbol{U}\cdot\nabla)\,A = (\boldsymbol{U}\cdot\nabla)\,B = 0, \quad B = (\boldsymbol{W}\cdot\nabla)\,A. \tag{7.5}$$

These equations imply that the streamlines of the flow $\boldsymbol{U}$ lie both on surfaces A = const. and on surfaces B = const., and are therefore the curves of intersection of these two families of surfaces. Each streamline is identified by the corresponding pair (A, B). The Bernoulli functions I, J defined by (5.6) are constant on these streamlines and are hence functions of A and B:

$$I = I(A, B), \quad J = J(A, B). \tag{7.6}$$

Consider now the first of (5.6):

$$\boldsymbol{U}\times\boldsymbol{W} = \nabla I(A, B) = I_A\,\nabla A + I_B\,\nabla B, \tag{7.7}$$

where the suffices A, B correspond to partial differentiation. Taking the vector product first with ∇A, then with ∇B, gives

$$B\boldsymbol{U} = I_B\,\nabla A\times\nabla B, \quad \{(\boldsymbol{W}\cdot\nabla)\,B\}\,\boldsymbol{U} = -I_A\,\nabla A\times\nabla B, \tag{7.8}$$

and hence

$$(\boldsymbol{W}\cdot\nabla)\,B = -I_A\,B/I_B. \tag{7.9}$$

Defining $\Psi(A, B)$ such that

$$\Psi_B = I_B/B, \tag{7.10}$$

equations (7.8) both become

$$\boldsymbol{U} = \nabla A\times\nabla\Psi, \tag{7.11}$$

and the pair $(A(\boldsymbol{x}), \Psi(\boldsymbol{x}))$ constitute the generalized stream-function of the flow $\boldsymbol{U}$.

We now consider the problem (2.1)–(2.5) together with (2.18)–(2.20) and (7.1)–(7.3). (Note that we do *not* here use the representation (3.18) for $\boldsymbol{u}$.) These equations still admit the energy integral E = const., and, for arbitrary $\Phi(a, b)$, we also have the integral

$$\int_{\mathscr{D}} \Phi(a, b)\,\mathrm{d}\tau = \text{const.}, \tag{7.12}$$

by virtue of (7.1), (7.3). Hence we may now construct the invariant functional

$$\tilde{R} \equiv \int_{\mathscr{D}} \{\tfrac{1}{2}(\boldsymbol{u}^2 + \boldsymbol{h}^2) + \Phi(a, b)\}\,\mathrm{d}\tau, \tag{7.13}$$

which has steady-state value

$$\tilde{R}_S = \int_{\mathscr{D}} \{\tfrac{1}{2}(\boldsymbol{U}^2 + \boldsymbol{H}^2) + \Phi(A, B)\}\,\mathrm{d}\tau. \tag{7.14}$$

Let us now regard $\boldsymbol{u}, \boldsymbol{h}, a$ and $\boldsymbol{m}$ as the independent variable fields which we may subject to smooth variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{h}$, δa, $\delta\boldsymbol{m}$ satisfying as usual

$$\nabla\cdot\delta\boldsymbol{u} = \nabla\cdot\delta\boldsymbol{h} = \nabla\cdot\delta\boldsymbol{m} = 0, \tag{7.15}$$

and $$\boldsymbol{n}\cdot\delta\boldsymbol{u} = \boldsymbol{n}\cdot\delta\boldsymbol{h} = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{7.16}$$

The fields $\boldsymbol{w}$ and b are now the dependent fields defined by (2.19) and (7.2), and with the first and second variations

$$\delta^1\boldsymbol{w} = \delta\boldsymbol{\omega}+\nabla\times(\boldsymbol{H}+\delta\boldsymbol{m}+\delta\boldsymbol{h}\times\boldsymbol{M}), \tag{7.17a}$$
$$\delta^2\boldsymbol{w} = \nabla\times(\delta\boldsymbol{h}\times\delta\boldsymbol{m}), \tag{7.17b}$$
$$\delta^1 b = (\boldsymbol{W}\cdot\nabla)\,\delta a+(\delta^1\boldsymbol{w}\cdot\nabla)\,A, \tag{7.17c}$$
$$\delta^2 b = (\delta^1\boldsymbol{w}\cdot\nabla)\,\delta a+(\delta^2\boldsymbol{w}\cdot\nabla)\,A. \tag{7.17d}$$

We now show that, with an appropriate choice of $\Phi(a,b)$, the steady flow (5.1), (5.6), (7.4) is a stationary point of the functional R. To this end, we calculate the first variation

$$\delta^1\tilde{R} = \int_{\mathscr{D}}\{\boldsymbol{U}\cdot\delta\boldsymbol{u}+\boldsymbol{H}\cdot\delta\boldsymbol{h}+\Phi_A\,\delta a+\Phi_B\,\delta^1 b\}\,\mathrm{d}\tau. \tag{7.18}$$

The term involving $\delta^1 b$ may be treated as follows:

$$\begin{aligned}\int_{\mathscr{D}}\Phi_B\,\delta^1 b\,\mathrm{d}\tau &= \int_{\mathscr{D}}\Phi_B[(\boldsymbol{W}\cdot\nabla)\,\delta a+(\nabla\times\delta^1\boldsymbol{w})\cdot\nabla A]\,\mathrm{d}\tau\\ &= \int_{\partial\mathscr{D}}\Phi_B(\boldsymbol{W}\delta a-\nabla A\times\delta^1\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S-\int_{\mathscr{D}}\delta a(\boldsymbol{W}\cdot\nabla)\,\Phi_B\,\mathrm{d}\tau+\int_{\mathscr{D}}\delta^1\boldsymbol{w}\cdot\boldsymbol{G}\,\mathrm{d}\tau,\end{aligned} \tag{7.19}$$

where $$\boldsymbol{G} = \nabla\times(\Phi_B\,\nabla A) = -\Phi_{BB}\,\nabla A\times\nabla B, \tag{7.20}$$

and $$\delta^1\boldsymbol{v} = \delta\boldsymbol{u}+\boldsymbol{H}\times\delta\boldsymbol{m}+\delta\boldsymbol{h}\times\boldsymbol{M}. \tag{7.21}$$

Hence, we obtain

$$\begin{aligned}\delta^1\tilde{R} &= \int_{\partial\mathscr{D}}\Phi_B(\boldsymbol{W}\delta a-\nabla A\times\delta^1\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S\\ &+\int_{\mathscr{D}}\{(\boldsymbol{U}+\boldsymbol{G})\cdot\delta\boldsymbol{u}+(\boldsymbol{H}-\boldsymbol{G}\times\boldsymbol{M})\cdot\delta\boldsymbol{h}+(\boldsymbol{G}\times\boldsymbol{H})\cdot\delta\boldsymbol{m}+[\Phi_A-(\boldsymbol{W}\cdot\nabla)\,\Phi_B]\,\delta a\}\,\mathrm{d}\tau.\end{aligned} \tag{7.22}$$

Here, the coefficient of $\delta\boldsymbol{u}$ in the volume integral is

$$\boldsymbol{U}+\boldsymbol{G} = B^{-1}(I_B-B\Phi_{BB})\,\nabla A\times\nabla B = B^{-1}F_B\,\nabla A\times\nabla B, \tag{7.23}$$

where $$F(A,B) \equiv \Phi+I-B\Phi_B = I-B^2\frac{\mathrm{d}}{\mathrm{d}B}\left(\frac{\Phi}{B}\right). \tag{7.24}$$

Moreover, the coefficient of δa in the volume integral is then

$$\Phi_A-B\Phi_{AB}+\frac{I_A B}{I_B}\Phi_{BB} = F_A-\frac{I_A}{I_B}F_B. \tag{7.25}$$

Hence if we choose Φ so that $F\equiv 0$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}B}\left(\frac{\Phi}{B}\right) = \frac{I}{B^2}, \tag{7.26}$$

then both coefficients (7.23) and (7.25) vanish.

With $\boldsymbol{G} = -\boldsymbol{U}$, the terms involving $\delta\boldsymbol{h}$ and $\delta\boldsymbol{m}$ in (7.22) become

$$\int\{(\boldsymbol{H}+\boldsymbol{U}\times\boldsymbol{M})\cdot\delta\boldsymbol{h}-(\boldsymbol{U}\times\boldsymbol{H})\cdot\delta\boldsymbol{m}\}\,\mathrm{d}\tau, \tag{7.27}$$

exactly as in (6.9), and these vanish for the same reason as in §6, using (5.6) and the conditions (7.15), (7.16) and $J = J_0$ (const.) on $\partial\mathscr{D}$.

Hence, finally, (7.22) reduces to the surface integral

$$\delta^1\tilde{R} = \int_{\partial\mathscr{D}} \Phi_B(W\delta a - \nabla A\times\delta^1\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{7.28}$$

We now identify three distinct circumstances in which this surface integral vanishes.

(i) Obviously $\delta^1\tilde{R} = 0$ if $\Phi_B = 0$ on $\partial\mathscr{D}$, and from (7.24) (with $F = 0$) this holds provided

$$\Phi(A,B) = -I(A,B) \quad \text{on} \quad \partial\mathscr{D}. \tag{7.29}$$

Now the condition

$$\boldsymbol{n}\cdot(\nabla A\times\nabla B) = 0 \quad \text{on} \quad \partial\mathscr{D} \tag{7.30}$$

implies that, provided $\boldsymbol{n}\times\nabla A \neq 0$, the fields A and B restricted to $\partial\mathscr{D}$ are functionally related, i.e.

$$B = B_S(A), \quad \text{say,} \quad \text{on} \quad \partial\mathscr{D}. \tag{7.31}$$

Moreover, it is evident from (7.26) that if $\Phi_0(A,B)$ is a solution of (7.26), then $\Phi_0 + Bg(A)$ is also a solution, for arbitrary $g(A)$. Provided $B \neq 0$ on $\partial\mathscr{D}$, this freedom allows us to satisfy (7.29) by choosing

$$g(A) = -(I(A,B)+\Phi_0(A,B))/B, \quad \text{with} \quad B = B_S(A). \tag{7.32}$$

Hence, provided

$$\boldsymbol{n}\times\nabla A \neq 0, \quad B \neq 0 \quad \text{for} \quad \boldsymbol{x}\in\partial\mathscr{D}, \tag{7.33}$$

we can always choose $\Phi(A,B)$ so that $\delta^1\tilde{R} = 0$.

(ii) The function $A(\boldsymbol{x})$ may be chosen so that $A = \text{const.}$ on $\partial\mathscr{D}$, (i.e. $\boldsymbol{n}\times\nabla A = 0$ at all points of $\partial\mathscr{D}$). We may suppose further that the variation $\delta a(\boldsymbol{x})$ of a satisfies the boundary condition $\delta a = 0$ on $\partial\mathscr{D}$ (so that the boundary value $a = A_0$ is maintained in the perturbed situation). Then obviously $\delta^1\tilde{R} = 0$ again.

(iii) The streamlines of $\boldsymbol{U}(\boldsymbol{x})$ on $\partial\mathscr{D}$ are closed curves on which $A = \text{const.}$ Let $\Gamma(A)$ be the circulation of $\boldsymbol{v}$ round $A = \text{const.}$ (which is conserved since $\boldsymbol{w} = \text{curl}\,\boldsymbol{v}$ is a frozen field). From (7.28)

$$\delta^1\tilde{R} = \int_{\partial\mathscr{D}} \Phi_B\,\delta a\,\boldsymbol{W}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{\partial\mathscr{D}} \Phi_B(\boldsymbol{n}\times\nabla A)\cdot\delta^1\boldsymbol{v}\,\mathrm{d}S. \tag{7.34}$$

Now $\Phi_B(A,B)$, with $B = B_S(A)$, is a function of A on $\partial\mathscr{D}$, and $\boldsymbol{n}\times\nabla A$ is parallel to $\boldsymbol{U}$ on $\partial\mathscr{D}$. If

$$\boldsymbol{n}\times\nabla A = f(A)\,\boldsymbol{U}/|\boldsymbol{U}|, \tag{7.35}$$

then the second integral may be treated by first integrating round curves $A = \text{const.}$, and then integrating over A. The first operation involves

$$\oint_{A=\text{const.}} \delta^1\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = \delta^1\Gamma(A). \tag{7.36}$$

Hence sufficient conditions for the vanishing of $\delta^1\tilde{R}$ are

$$\boldsymbol{W}\cdot\boldsymbol{n} = 0, \quad |\boldsymbol{n}\times\nabla A| = f(A), \quad \delta^1\Gamma(A) = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{7.37}$$

These conditions are convenient when the flow is either two-dimensional, or axisymmetric, or has some comparable symmetry.

For $\delta^2 \tilde{R}$ we have

$$\delta^2 \tilde{R} = \int_{\mathscr{D}} [\delta u_i \, \delta u_i + \delta h_i \, \delta h_i + \Phi_{AA}(\delta a)^2 + \Phi_{BB}(\delta^1 b)^2 + 2\Phi_{AB} \, \delta a \, \delta^1 b + 2\Phi_B \, \delta^2 b] \, \mathrm{d}\tau, \quad (7.38)$$

where $\delta^1 \boldsymbol{w}$, $\delta^2 \boldsymbol{w}$, $\delta^1 b$, $\delta^2 b$ are presented in the form (7.17). By simple operations and using (7.26) we obtain

$$2\delta^2 \tilde{R} = \int_{\mathscr{D}} \left\{ \frac{1}{BI_B}(I_A \, \delta a + I_B \, \delta^1 b)^2 - \frac{2}{B} \delta a (\delta^1 \boldsymbol{w} \cdot \boldsymbol{\nabla}) I + U(\delta \boldsymbol{m} \times \delta \boldsymbol{h}) \right\} \mathrm{d}\tau$$
$$+ \int_{\partial \mathscr{D}} \{ 2\Phi_B [\delta a \delta^1 \boldsymbol{w} + \delta \boldsymbol{m} (\delta \boldsymbol{h} \cdot \boldsymbol{\nabla}) A] + \frac{\Phi_A}{B} W (\delta a)^2 \} \cdot \boldsymbol{n} \, \mathrm{d}S. \quad (7.39)$$

The flow is stable if $\delta^2 \tilde{R}$ is definite in sign for arbitrary choice of $\delta \boldsymbol{u}$, $\delta \boldsymbol{h}$, δa, $\delta \boldsymbol{m}$ with $\delta^1 \boldsymbol{w}$, $\delta^1 b$ given by (7.17*a*, *c*).

8. Conclusions

Starting with the identification of the frozen fields $(\boldsymbol{h}, \boldsymbol{w})$ of ideal magnetohydrodynamics, we have given a systematic development of the transformation properties of the MHD equations (thus generalizing the Weber transformation of the Euler equations) and of the underlying Hamiltonian structure of these equations. We have then considered steady-state properties, and have constructed energy variational principles characterizing the steady states. In a subsequent paper, we shall discuss the application of these principles to flows and fields with particular symmetries.

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